Is Free Choice Permission Admissible in Classical Deontic Logic?

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Abstract

We explore how, and if, free choice permission (**FCP**) can be accepted when we consider deontic conflicts between certain types of permissions and obligations. **FCP** can license, under some minimal conditions, the derivation of an indefinite number of permissions. We discuss this and other drawbacks and present four Hilbert-style classical deontic systems admitting a guarded version of **FCP**. The systems that we present are not too weak from the inferential viewpoint, as far as permission is concerned, and do not commit to weakening any specific logic for obligations.

Keywords: Free Choice Permission, Weak and Strong Permission

1 Introduction and Background

A significant part of the literature in deontic logic revolves around the discussions of puzzles and paradoxes which show that certain logical systems are not acceptable—typically, this happens with deontic **KD**, i.e., Standard Deontic Logic (**SDL**)—or which suggest that obligations and permissions should enjoy some desirable properties.

One well-known puzzle is the the so-called Free Choice Permission paradox, which was originated by the following remark by von Wright in [24, p. 21]:

> "On an ordinary understanding of the phrase 'it is permitted that', the formula '$\mathbf{P}(p \vee q)$' seems to entail '$\mathbf{P}p \wedge \mathbf{P}q$'. If I say to somebody 'you may work or relax' I normally mean that the person addressed has my permission to work and also my permission to relax. It is up to him to choose between the two alternatives."

Usually, this intuition is formalised by the following schema:

$$\mathbf{P}(p \vee q) \rightarrow (\mathbf{P}p \wedge \mathbf{P}q) \qquad \textbf{(FCP)}$$

Many problems have been discussed in the literature around **FCP**: for a comprehensive overview, discussion, and some solutions, see [15, 11, 22].

Three basic difficulties can be identified, among the others [11, p. 43]:

- **Problem 1: Permission Explosion Problem** – "That if anything is permissible, then everything is, and thus it would also be a theorem that nothing is obligatory," [22], for example "If you may order a soup, then it is not true that you ought to pay the bill" [6];
- **Problem 2: Closure under Logical Equivalence Problem** – "In its classical form **FCP** entails that classically equivalent formulas can be substituted to the scope of a permission operator. This is also implausible: It is permitted to eat an apple or not iff it is permitted to sell a house or not";
- **Problem 3: Resource Sensitivity Problem** – "Many deontic logics become resource-insensitive in the presence of **FCP**. They validate inferences of the form 'if the patient with stomach trouble is allowed to eat one cookie then he is allowed to eat more than one'".

We focus on another basic problem: how, and if, **FCP** can be accepted when we have incompatibilities between certain varieties of permissions and prohibitions/obligations. The issue is that since Problem 1 licenses the derivation that anything is permitted provided that something is permitted, no prohibition/obligation is allowed, otherwise we get an inconsistency [22]. In doing so, we offer simple logics that take two of the three problems above into account.

The layout of the paper is as follows. The remainder of this section briefly comments on the three major problems mentioned above: the Permission Explosion Problem (Section 1.1), the Closure under Logical Equivalence Problem (Section 1.2), and the Resource Sensitivity Problem (1.3). Section 2 illustrates the theoretical intuitions and assumptions that we adopt to analyse free choice permission. In particular, we assume the distinction between norms and obligations/permissions, and we study the role of deontic incompatibilities, the duality principle, and why free choice permission is strong permission. Section 3 quickly reviews some work that have direct implications for our proposal. Finally, Section 4 presents some minimal deontic systems, four Hilbert-style deontic systems admitting guarded variants of **FCP**: the systems that we present are not too weak from the inferential viewpoint, as far as permission is concerned, and do not commit to weakening any specific logic for obligations. Some conclusions end the paper. An appendix offers proofs of the formal properties of the proposed systems presented in Section 4.

1.1 Problem 1: Permission Explosion Problem

One of the most acute problems springing from **FCP** is obtained in **SDL**, where, if at least one obligation $\mathbf{O}p$ is true, then by necessitation and propositional logic, we get $\mathbf{O}(p \vee q)$. Since axiom **D** is in **SDL**, i.e $\mathbf{O}p \rightarrow \neg\mathbf{O}\neg p$ is valid, we trivially obtain $\neg\mathbf{O}\neg(p \vee q)$, thus, assuming the Duality principle

$$\mathbf{P} =_{def} \neg\mathbf{O}\neg \qquad (\mathbf{Duality})$$

we derive through **FCP** that $\mathbf{P}q$. Hence, **SDL** licenses that, if something is obligatory, then everything is permitted.

However, a careful analysis shows that this undesired result is not strictly due to **SDL** as such, but to adopting any monotonic modal deontic logic [10], i.e. any system just equipped with inference rule **RM**:

$$\frac{\vdash p \rightarrow q}{\vdash \mathbf{O}p \rightarrow \mathbf{O}q}. \tag{\textbf{RM}}$$

or, alternatively with

$$\frac{\vdash p \equiv q}{\vdash \mathbf{O}p \equiv \mathbf{O}q}. \tag{\textbf{RE}}$$

plus the following axiom schema

$$\mathbf{O}(a \wedge b) \rightarrow (\mathbf{O}a \wedge \mathbf{O}b). \tag{\textbf{M}}$$

Indeed, assume Classical Propositional Logic (**CPL**), **FCP**, and **RM** for **P** [1] and consider the following derivation:

$$\begin{array}{ll} 1.\ p \rightarrow (p \vee q) & \mathbf{CPL} \\ 2.\ \mathbf{P}p \rightarrow \mathbf{P}(p \vee q) & 1, \mathbf{RM} \\ 3.\ \mathbf{P}p \rightarrow (\mathbf{P}p \wedge \mathbf{P}q) & 2, \mathbf{FCP}, \mathbf{CPL} \\ 4.\ \mathbf{P}p \rightarrow \mathbf{P}q & 3, \mathbf{CPL} \end{array}$$

In this context, it is enough if we have that $\mathbf{P}p$ is true to derive that any other permission $\mathbf{P}q$, i.e., $\mathbf{P}p \vdash \mathbf{P}q$ for any p, q. Whenever **FCP** is accepted, such a problem strictly depends on the characteristic schemas and inference rules of monotonic modal logics, as the above derivation—or a simple semantic analysis—shows. Hence, permission explosion is not a problem of **SDL**, but of any weaker modal deontic logic which is at least closed under classical implication or which is closed under logical equivalence and allows for the distribution of **P** over implication. Notice that **Duality** plays no substantial role. Accordingly, we can have that **RM** is valid for permission, if **P** and **O** are duals and the logic for **O** is a monotonic modal logic, or **P** is independent of **O** and **RM** is assumed for **P**.

In conclusion, if we want not to completely reject the intuition behind **FCP**, we have two non-exclusive options to be explored in order to avoid the Permission Explosion Problem:

No-CPL: abandon **CPL** and adopt suitable non-classical logical connectives;

No-RM: abandon inference rule **RM** (or schema **M**) and endorse very weak modal logics (i.e., the classical ones [10, chap. 8]). [2]

Our paper aims at exploring under what conditions **No-CPL** can be avoided by accepting at least a restricted version of **FCP**. Hence, it seems that **No-RM** thesis must be accepted.

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[1] With **RM-P** we mean the inference rule $\vdash p \rightarrow q / \vdash \mathbf{P}p \rightarrow \mathbf{P}q$. Indeed, it is standard result that every system closed under **RM** for an operator is closed under the rule of the dual of the operator [cf. 10, p. 238–239, 243]. We will use **RM** to refer in general to the rule $\vdash p \rightarrow q / \vdash \Box p \rightarrow \Box q$ for any modal operator $\Box$.

[2] We state in Section 1.2 why it is convenient not to drop **RE**.

1.2 Problem 2: Closure under Logical Equivalence Problem

In the previous section we mentioned that **RM** must be weakened. Hence, we can also drop **RE** and keep axiom schema **M**. This choice could look satisfactory for those who consider problematic the fact that the logic for **P** is closed under logical equivalence.

We take here another route. Incidentally, one can argue that the implausibility of "It is permitted to eat an apple or not iff it is permitted to sell a house or not" does not depend on **RE**, but rather on the fact that "It is permitted to eat an apple or not" is $\mathbf{P}\top$, which looks quite odd. However, besides this problem—which would lead us to commit to specific philosophical views—dropping **RE** has in general two controversial technical side effects:

- it rejects standard semantics for modal logics, since the class of all neighbourhood frames validate **RE**: [10] argued in fact that classical systems (i.e., containing **RE** but not **RM**) are the minimal modal logics;
- it fails to make, for instance, $\mathbf{O}p$ and $\mathbf{P}\neg p$ logically incompatible under the Duality Principle (while $\mathbf{O}p$ and $\neg\mathbf{O}p$ of course are); similarly, $\mathbf{O}\neg p$ and $\mathbf{O}\neg\neg p$, or $\mathbf{O}(p \vee q)$ and $\mathbf{O}(\neg p \wedge \neg q)$, are not incompatible too (while they of course should be).

In conclusion, we standardly assume that **RE** holds both for permissions and obligations, which means that any logic for free choice permission must be a *classical system of deontic logic* in [10]'s sense, i.e., any modal deontic logic closed under logical equivalence and not under logical consequence.

1.3 Problem 3: Resource Sensitivity Problem

It has been noted [19] that from "You may eat an apple or a pear", one can infer "You may eat an apple and that You may eat a pear", but not "You may eat an apple and a pear" [7, p. 2].

We simply observe that the systems proposed in Section 4 do not license in general the inference above. However, a thoughtful treatment of this problem—the Resource Sensitivity Problem—goes beyond the scope of this paper. In fact, it has been widely discussed in the literature that it is strictly related to considerations from action theory, which have often found solutions shifting from **CPL** to non-classical logics such as the substructural ones [see, among others, 7, 4, 11]. In conclusion, we do not commit here to find any suitable solution to such a problem.

2 Three Basic Intuitions

We are going to present some deontic systems that accommodate restricted variants of **FCP**. This is done under some minimal philosophical assumptions, which can in principle be compatible with several deontic theories. In this section, we illustrate such fundamental intuitions and assumptions.

2.1 The Distinction between Norms and Obligations

We assume in the background a conceptual distinction between norms, on one side, and obligations and permissions, on the other side. The general idea of norms is that they describe conditions under which some behaviours are deemed as 'legal'. In the simplest case, a behaviour can be qualified by an obligation (or a prohibition, or a permission), but often norms additionally specify the consequences of not complying with them, and what sanctions follow from violations and whether such sanctions compensate for the violations. The scintilla for this idea is the very influential contribution [1], which is complementary to the (modal) logic-based approaches to deontic logic. The key feature of this approach is that norms are dyadic constructs connecting applicability conditions to a deontic consequence. A large number of such pairs would constitute an interconnected system called a *normative system* [for more recent proposals in this direction, see 20, 21, 14, 12].

To be clear, *this paper does not present any logic of norms, but our proposal for a logic of obligations and permissions—with restricted variants of **FCP**—can be better understood if one keeps in mind some intuitions about how norms should logically behave* and about the relation between the logic of norms and deontic logic. In particular, our assumptions are:

- obligations and permissions exist because norms generate them when applicable;
- once obligations and permissions are generated from norms—which requires us to reason about norms—we can still perform some reasoning with the resulting obligations and permissions—this is the task of *deontic logic in a strict sense*, i.e., the logic of obligations and permissions;
- norms can be in conflict—without being inconsistent— but this does not hold for obligations and permissions.

Hence, we distinguish two levels of analysis: a *norm-logic level* and a resulting *deontic-logic level*. This paper only technically deals with the second level of analysis.

Assume for example that we have two norms $n_1 : p \Rightarrow \mathbf{O}\neg q$ and $n_2 : p \Rightarrow \mathbf{P}q$, where $\Rightarrow$ is any if-then suitable logical relation connecting applicability conditions of norms and their deontic effects. We can indeed have them—for example, in a legal system—but the point is what obligations/permissions we can obtain from them. A rather standard assumption is that in order to correctly derive deontic conclusions we need to solve the conflict between n_1 and n_2. Specifically, our general view is prudent (or skeptical, as one says in non-monotonic logics), because, unless we know how to solve the conflict (typically, by establishing that n_1 is stronger than n_2 or vice versa), we do not know if $\mathbf{O}\neg q$ or $\mathbf{P}q$ holds. Since we do not accept that both can hold, it is pointless to consider at the deontic level that $\mathbf{O}\neg q$ *and* $\mathbf{P}q$ are true—while any logic of norms can have both n_1 and n_2.

In conclusion, we impose deontic consistency at the deontic-logic level, i.e.,

$\mathbf{O}p \wedge \mathbf{O}\neg p \rightarrow \bot$.

2.2 Deontic Incompatibilities, Duality, and FCP

With the above said, the issue is whether **FCP** is an appropriate principle to adopt for normative reasoning. Our view is that this principle in general is not, even when Problem 1 and 2 above are solved. We provide below a simple counterexample to it, which considers the interplay between free choice permissions and prohibitions.

Example 2.1 When you have dinner with guests the etiquette allows you to eat or to have a conversation with your fellow guests. However, it is forbidden to speak while eating.

The full representation of the example is that each choice is permitted when one refrains from exercising the other one. In a situation when one eats, there is the prohibition to speak, while when one speaks, there is the prohibition to eat. Hence, it means that we can detach any single permission only if the content of such permission is not forbidden. Given that Example 2.1 provides a counterexample to **FCP**, the question is whether we want to derive the individual permissions when one of the two disjuncts holds and we already satisfy the disjunctive permission. The reason is that the individual permissions, each on its own, can trigger other obligations or permissions. The following example illustrates this scenario.

Example 2.2 Suppose a shop has the following policy for clothes bought online. If the size of an item is not a perfect fit, then the customer is entitled to either exchange the item for free or to keep the item and receive a 10$ refund. However, customers electing to keep the item are not entitled to the refund, and customers opting for the refund are not entitled to exchange the item for free. Furthermore, customers who elect to exchange the item (when entitled to do so) have to return it with the original package.

The example can be formalised as follows:

$$\begin{gathered} \mathit{online} \wedge \neg \mathit{fit} \rightarrow \mathbf{P}(\mathit{exchange} \vee \mathit{refund}) \\ \mathit{exchange} \rightarrow \mathbf{O}\neg \mathit{refund} \\ \mathit{refund} \rightarrow \mathbf{O}\neg \mathit{exchange} \\ \mathbf{P}\mathit{exchange} \wedge \mathit{exchange} \rightarrow \mathbf{O}\mathit{original} \end{gathered}$$

Suppose that a customer elects to exchange an item bought online that is not a perfect fit instead of asking for the refund. Intuitively, given that we cannot derive that exchanging is not forbidden ($\mathbf{O}\neg \mathit{exchange}$) at least the weak permission of exchanging the item should hold. However, in a deontic logic without **FCP** (or a restricted version of it) we are not able to derive the permission, and then we are not able to derive other obligations or permissions depending on it: in the example, the obligation to return the item with the original package.

We will return in Section 2.3 to the logical import of the above scenarios in a classical system of deontic logic. For the moment, taking stock of the examples we just notice that **FCP** could be reformulated as follows:

$$(\mathbf{P}(p \vee q) \wedge (\neg\mathbf{O}\neg p \wedge \neg\mathbf{O}\neg q)) \rightarrow (\mathbf{P}p \wedge \mathbf{P}q). \tag{1}$$

However, assuming **Duality**, $\neg\mathbf{O}\neg p$ is equivalent to $\mathbf{P}p$, thus (1) reduces to

$$(\mathbf{P}(p \vee q) \wedge (\mathbf{P}p \wedge \mathbf{P}q)) \rightarrow (\mathbf{P}p \wedge \mathbf{P}q). \tag{2}$$

(2) is a propositional tautology. Thus, (1) does not extend the expressive power of the logic unless one assumes a logic where obligation and permission are not the duals.

2.3 Strong Permission, Classical Systems, and FCP

When permission is no longer the dual of obligation, we enter the territory of strong permission [25, 3, 2] [3] . As is well-known, while it is *sufficient* to show that $\mathbf{O}\neg p$ is *not* the case to argue that p is weakly permitted, this does not hold for strong permission, for which the normative system *explicitly* says that there exists at least one norm permitting p [2, p. 353–355].

In order to keep track of these two cases at the deontic-logic level, we can standardly distinguish in the deontic language two permission operators, $\mathbf{P_w}$ for weak permission (such that $\mathbf{P_w}p =_{def} \neg\mathbf{O}\neg p$) and $\mathbf{P_s}$ for strong permission (where **Duality** does not hold).

What is the minimal logic of strong permission at the deontic level in which some reasonable version of free choice permission can be accepted?

We mentioned that **RM** must be rejected. In fact, besides the Permission Explosion Problem, one may also argue that it is reasonable not to derive $\mathbf{P_s}(p \vee q)$ from any $\mathbf{P_s}p$ because we could have in the background that the normative system consists just of an explicit norm $a \Rightarrow \mathbf{P_s}p$. If we have that, in presence of some version of free choice permission, you may also detach $\mathbf{P_s}q$, which is against the above-mentioned intuition that the strong permission should follow from explicit norms, or from combinations of them in normative systems where all disjuncts are explicitly considered [see, e.g., the discussion in 2, p. 354–355].

Second, as said above, deontic consistency should be ensured:

$$\mathbf{O}p \wedge \mathbf{P_s}\neg p \rightarrow \bot \tag{$\mathbf{D_s}$}$$
$$\mathbf{O}p \wedge \mathbf{O}\neg p \rightarrow \bot \tag{$\mathbf{D_w}$}$$

Notice that $\mathbf{D_w}$ is the standard **D** axiom of Standard Deontic Logic establishing the so called *external consistency* of obligations that, in turn, implies consistency among obligations and (weak) permissions. From $\mathbf{D_s}$ we obtain, as expected,

[3] Besides von Wright's theory [25], there is another sense in the literature of strong permission [16].

that strong permission entails weak permission [see, e.g., 2, p. 354], but not the other way around:

$$\mathbf{P_s}p \rightarrow \mathbf{P_w}p.$$

This is reasonable because the fact that at the norm-level we derive that p is permitted using an explicit permissive norm n means that no prohibitive norm n' (forbidding p) successfully applies or prevails over n.

What about free choice permission? Coupling Assumptions 1 and 2 with the distinction between weak and strong permission allows us to identify a guarded variant of **FCP** for strong permission, consisting of two schemata:

$$(\mathbf{P_s}(p \vee q) \wedge \mathbf{O}\neg p) \rightarrow \mathbf{P_s}q \qquad (\mathbf{FCP_O})$$
$$(\mathbf{P_s}(p \vee q) \wedge \mathbf{P_w}p \wedge \mathbf{P_w}q) \rightarrow (\mathbf{P_s}p \wedge \mathbf{P_s}q) \qquad (\mathbf{FCP_P})$$

These schemata take stock of what we said: you can detach from a disjunctive strong permission any single strong permission only if this last is weakly permitted.

The idea of the combination of the two axioms is that from repeated applications of $\mathbf{FCP_O}$ and from a disjunctive permission, we can obtain the maximal sub-disjunction such that no element is forbidden, and then, the application of the $\mathbf{FCP_P}$ allows us to derive the individual strong permissions that are not forbidden. Notice that we cannot assume the following formula as the axiom for free choice permission.

$$\mathbf{P_s}\left(\bigvee_{i=1}^{n} p_i\right) \wedge \left(\bigwedge_{j=1}^{m<n} \mathbf{O}\neg p_j\right) \rightarrow \bigwedge_{k=m+1}^{n} \mathbf{P_s}p_k$$

The problem is that we do not know in advance how many elements of the disjunctive permission are (individually) forbidden. Consider for example, a theory consisting of the following formulas:

$$\mathbf{P_s}(p \vee q \vee r \vee s \vee t) \qquad \mathbf{O}\neg p \qquad \mathbf{O}\neg q \qquad \mathbf{O}\neg r$$

Here, one could use the conjunction $\mathbf{O}\neg p \wedge \mathbf{O}\neg q$ to obtain $\mathbf{P_s}r$, $\mathbf{P_s}s$ and $\mathbf{P_s}t$, but then we have a contradiction from $\mathbf{P_s}r$ and $\mathbf{O}\neg r$ (from axiom $\mathbf{D_s}$). Notice, that in general, we are not able to use $\mathbf{FCP_O}$ to detach a single (strong) permission, but a disjunction corresponding to the "remainder" of the disjunction, that is, in the case above, $\mathbf{P_s}(s \vee t)$. Then, we can use the $\mathbf{FCP_P}$ to "lift" the remaining elements from weak permissions to strong permissions. The only case when we can obtain an individual strong permission from a permissive disjunction is when the remainder is a singleton; but this means, that all the other elements of the permissive disjunction were forbidden. This further means that a disjunctive strong permission holds if at least one of its elements can be legally exercised. Going back to the example, if one extends the theory with $\mathbf{O}\neg s$, then we can derive $\mathbf{P_s}t$.

Consider the situation described in Example 2.1. The scenario can be formalised as follows (where e and s stand for "to eat" and "to speak"):

$$\mathbf{P_s}(e \vee s)$$
$$s \rightarrow \mathbf{O}\neg e$$
$$e \rightarrow \mathbf{O}\neg s$$

In a logic endorsing the unrestricted version of free choice permission, we have $\mathbf{P_s}e$ and $\mathbf{P_s}s$. This means that as soon as one exercises one of the choices, we get that the other choice is at the same time permitted and forbidden, a situation that is either paradoxical or contradictory. Thus, the only way to avoid this kind of conflict is to refrain from exercising any of the two choices. However, this means that one is not really free to choose between the two options. Accordingly, either one has to adopt a restricted version of the free choice permission or abandon it. Notice, that axiom $\mathbf{FCP_O}$ allows us to conclude that given e, s is forbidden ($\mathbf{O}\neg s$), and thus that e is permitted ($\mathbf{P_s}e$); similarly, one gets $\mathbf{P_s}s$ from s, which implies $\mathbf{O}\neg e$. Similarly, for Example 2.2 when we formalise it using strong permission $\mathbf{P_s}$ instead of $\mathbf{P}$, Axiom $\mathbf{FCP_O}$ allows us to derive $\mathbf{P_s}\mathit{exchange}$ from which we can conclude $\mathbf{O}\mathit{original}$.

Consider $\mathbf{FCP_P}$. One may argue why, in symmetry with $\mathbf{FCP_O}$, we cannot rather have

$$(\mathbf{P_s}(p \vee q) \wedge \mathbf{P_w}p) \rightarrow \mathbf{P_s}p \qquad (\mathbf{FCP2_P})$$

Technically, it is obvious that $\mathbf{FCP2_P}$ implies $\mathbf{FCP_P}$ but not the other way around, so both options are available. The variant $\mathbf{FCP_P}$ is more prudent in that it licenses the detachment of an individual strong permission *only if* the normative system explicitly deals with that specific disjunct, while the second allows for the derivation in a slightly more relaxed way. So, if one wants to strictly reframe the structure of standard **FCP** in a guarded version but does not want $\mathbf{FCP2_P}$, then $\mathbf{FCP_P}$ is the right option.

We should notice that the above schemata for free choice permission do not necessarily require the technical idea of deontic consistency, unless we assume—but we don't—that obligation implies strong permission, and despite the fact that the consistency problem can occur if we endorse $\mathbf{D_s}$—as we do—and so that strong permission implies weak permission.

3 Related Works

Most of the work on the development of logical systems related to the problem of Free Choice Permission concentrate on logics accepting the **FCP** principle. Some work focus on the resource aspects and propose the use of substructural logics to address the problem, see for example [7]. Similarly to our work, in the sense of a non-normal deontic logic, is the proposal in [6, 5]. In fact, even though they have a different philosophical backgrounds based on the of open reading of permissions [17, 9], they propose *simple non-normal axiomatisations for obligation and permission*—as we do—which avoid, e.g., Problem 1 and which are based on the concept of free choice permission as strong permission or, anyway, as a type of permission without **Duality**.

The scenario in Example 2.2 indicates that Deontic Logic should accept **FCP**, but at the same time Example 2.1 points out that it cannot accept in an unrestricted form. In this regard, the proposal by Asher and Bonevac [6] shares with us the idea of limiting the applicability of **FCP**. Their solution is based on a deontic logic taking a non-monotonic logic as the underlying reasoning mechanism instead classical propositional logic as we do. Accordingly, in their system instances of **FCP** are derivable unless they are defeated. In addition, their logic is not closed under logical equivalence.

4 Four Minimal Deontic Axiomatisations with Guarded Free Choice Permission

Finally, we present some minimal deontic systems, four Hilbert-style deontic systems admitting a guarded version of **FCP**. The systems that we present are not too weak from the inferential viewpoint, as far as permission is concerned, and do not commit to weakening any specific logic for obligations.

4.1 Language, Axioms and Inference Rules

The modal language and the concept of well formed formula are defined as usual [see 10, 8]. We just recall that we have three modal operators, two $\Box$ operators, $\mathbf{O}$ for obligations and $\mathbf{P_s}$ for strong permissions, and $\mathbf{P_w}$ for weak permission. As usual, we assume $\mathbf{P_w}$ to be an abbreviation for $\neg\mathbf{O}\neg$.

For convenience, let us synoptically recall below all relevant schemata and inference rules, where $\Box \in \{\mathbf{O}, \mathbf{P_s}\}$.

Inference Rules:

$\mathbf{RE} := \vdash A \equiv B \;\Rightarrow\; \vdash \Box A \leftrightarrow \Box B$

$\mathbf{RM} := \vdash A \to B \;\Rightarrow\; \vdash \Box A \to \Box B$

Schemata:

$\mathbf{M} :=\; \Box(p \wedge q) \to (\Box p \wedge \Box q)$

$\mathbf{FCP_O} :=\; (\mathbf{P_s}(p \vee q) \wedge \mathbf{O}\neg p) \to \mathbf{P_s} q$

$\mathbf{FCP_P} :=\; (\mathbf{P_s}(p \vee q) \wedge \mathbf{P_w} p \wedge \mathbf{P_w} q) \to (\mathbf{P_s} p \wedge \mathbf{P_s} q)$

$\mathbf{FCP2_P} :=\; (\mathbf{P_s}(p \vee q) \wedge \mathbf{P_w} p) \to \mathbf{P_s} p$

$\mathbf{D_s} :=\; \mathbf{O}p \wedge \mathbf{P_s}\neg p \to \bot$

$\mathbf{D_w} :=\; \mathbf{O}p \wedge \mathbf{P_w}\neg p \to \bot$

$\mathbf{P_s P_w} :=\; \mathbf{P_s} p \to \mathbf{P_w} p.$

Given the discussion of Section 2, we can identify some deontic systems, as specified in Table 1. Notice that we consider also systems $\mathbf{FCP_2}$ and $\mathbf{FCP_4}$, which are monotonic, so they contain **RM**. Strictly speaking, this is the limit which we cannot trespass, since we have restricted forms of Permission Explosion. We will return on this in the concluding section of the paper.

Deontic System	Properties	Derivable
$\mathbf{E} := \mathbf{RE}$		
$\mathbf{Min} := \mathbf{RE} \oplus \mathbf{D_s} \oplus \mathbf{D_w}$		$\mathbf{P_sP_w}$
$\mathbf{FCP_1} := \mathbf{Min} \oplus \mathbf{FCP_O} \oplus \mathbf{FCP_P}$		$\mathbf{P_sP_w}$
$\mathbf{FCP_2} := \mathbf{FCP_1} \oplus \mathbf{M}$	$\mathbf{FCP_1} \subset \mathbf{FCP_2}$	$\mathbf{P_sP_w}$
$\mathbf{FCP_3} := \mathbf{Min} \oplus \mathbf{FCP_O} \oplus \mathbf{FCP2_P}$	$\mathbf{FCP_1} \subset \mathbf{FCP_3}$	$\mathbf{P_sP_w}$, $\mathbf{FCP_P}$
$\mathbf{FCP_4} := \mathbf{FCP_3} \oplus \mathbf{M}$	$\mathbf{FCP_2} \subset \mathbf{FCP_4}$ $\mathbf{FCP_3} \subset \mathbf{FCP_4}$	$\mathbf{P_sP_w}$, $\mathbf{FCP_O}$ $\mathbf{FCP_P}$, $\mathbf{FCP2_P}$

Table 1
Deontic Systems

4.2 Semantics and System Properties

Let us begin with standard concepts. Assume that PROP is the set of atomic sentences.

Definition 4.1 A *deontic neighbourhood frame* $\mathcal{F}$ is a structure $\langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P} \rangle$ where

- W is a non-empty set of possible worlds;
- $\mathcal{N}_\mathbf{O}$ and $\mathcal{N}_\mathbf{P}$ are functions $W \mapsto 2^{2^W}$.

Definition 4.2 A *deontic neighbourhood model* $\mathcal{M}$ is a structure $\langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P}, V \rangle$ where $\langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P} \rangle$ is a deontic neighbourhood frame and V is an evaluation function PROP $\mapsto 2^W$.

Definition 4.3 [Truth in a model] Let $\mathcal{M}$ be a model $\langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P}, V \rangle$ and $w \in W$. The truth of any formula p in $\mathcal{M}$ is defined inductively as follows:

(i) standard valuation conditions for the boolean connectives;

(ii) $\mathcal{M}, w \models \mathbf{O}p$ iff $||p||_\mathcal{M} \in \mathcal{N}_\mathbf{O}(w)$,

(iii) $\mathcal{M}, w \models \mathbf{P_s}p$ iff $||p||_\mathcal{M} \in \mathcal{N}_\mathbf{P}(w)$,

(iv) $\mathcal{M}, w \models \mathbf{P_w}p$ iff $W - ||p||_\mathcal{M} \notin \mathcal{N}_\mathbf{O}(w)$,

where, as usual, $||p||_\mathcal{M}$ is the truth set of p wrt to $\mathcal{M}$

$$||p||_\mathcal{M} = \{w \in W : \mathcal{M}, w \models p\}.$$

A formula p is *true at a world* in a model iff $\mathcal{M}, w \models p$; *true in a model* $\mathcal{M}$, written $\mathcal{M} \models p$ iff for all worlds $w \in W$, $\mathcal{M}, w \models p$; *valid in a frame* $\mathcal{F}$, written $\mathcal{F} \models p$ iff it is true in all models based on that frame; *valid in a class* $\mathcal{C}$ *of frames*, written $\mathcal{C} \models p$, iff it is valid in all frames in the class. An inference rule $P_1, \dots P_n \Rightarrow C$ (where $P_1, \dots P_n$ are the premises and C the conclusion) is valid in a class $\mathcal{C}$ of frames iff, for any $\mathcal{F} \in \mathcal{C}$, if $\mathcal{F} \models P_1, \dots, \mathcal{F} \models P_n$ then $\mathcal{F} \models C$ [4].

We can now characterise different classes of deontic neighbourhood frames that are adequate of the deontic systems in Table 1.

[4] Of course, if any P_k has the form $\vdash p$ then $\mathcal{F} \models P_1$ trivially means $\mathcal{F} \models p$.

Definition 4.4 [Frame Properties] Let $\mathcal{F} = \langle W, \mathcal{N}_{\mathbf{O}}, \mathcal{N}_{\mathbf{P}} \rangle$ be a deontic neighbourhood frame.

- $\Box$-***supplementation***: $\mathcal{F}$ is $\Box$-*supplemented*, $\Box \in \{\mathbf{O}, \mathbf{P}\}$, iff for any $w \in W$ and $X, Y \subseteq W$, $X \cap Y \in \mathcal{N}_{\Box}(w) \Rightarrow X \in \mathcal{N}_{\Box}(w) \,\&\, Y \in \mathcal{N}_{\Box}(w)$;
- $\mathbf{P_w}$-***coherence***: $\mathcal{F}$ is $\mathbf{P_w}$-*coherent* iff for any $w \in W$ and $X \subseteq W$, $X \in \mathcal{N}_{\mathbf{O}}(w) \Rightarrow W - X \notin \mathcal{N}_{\mathbf{O}}(w)$;
- $\mathbf{P_s}$-***coherence***: $\mathcal{F}$ is $\mathbf{P_s}$-*coherent* iff for any $w \in W$ and $X \subseteq W$, $X \in \mathcal{N}_{\mathbf{P}}(w) \Rightarrow W - X \notin \mathcal{N}_{\mathbf{O}}(w)$;
- $\mathbf{FCP_O}$-***permission***: $\mathcal{F}$ is $\mathbf{FCP_O}$-*permitted* iff for any $w \in W$ and $X, Y \subseteq W$, $X \cup Y \in \mathcal{N}_{\mathbf{P}}(w) \,\&\, W - Y \in \mathcal{N}_{\mathbf{O}}(w) \Rightarrow X \in \mathcal{N}_{\mathbf{P}}(w)$;
- $\mathbf{FCP_P}$-***permission***: $\mathcal{F}$ is $\mathbf{FCP_P}$-*permitted* iff for any $w \in W$ and $X, Y \subseteq W$, $X \cup Y \in \mathcal{N}_{\mathbf{P}}(w) \,\&\, W - X \notin \mathcal{N}_{\mathbf{O}}(w) \,\&\, W - Y \notin \mathcal{N}_{\mathbf{O}}(w) \Rightarrow X \in \mathcal{N}_{\mathbf{P}}(w) \,\&\, X \in \mathcal{N}_{\mathbf{P}}(w)$;
- $\mathbf{FCP2_P}$-***permission***: $\mathcal{F}$ is $\mathbf{FCP2_P}$-*permitted* iff for any $w \in W$ and $X, Y \subseteq W$, $X \cup Y \in \mathcal{N}_{\mathbf{P}}(w) \,\&\, W - X \notin \mathcal{N}_{\mathbf{O}}(w) \Rightarrow X \in \mathcal{N}_{\mathbf{P}}(w)$;

Below are some relevant characterisation results. All the proofs for this section are in the Appendix.

Lemma 4.5 *For any deontic neighbourhood frame* $\mathcal{F}$*,*

(i) $\mathbf{D_s}$ *is valid in the class of* $\mathbf{P_s}$-coherent *frames;*
(ii) $\mathbf{D_w}$ *is valid in the class of* $\mathbf{P_w}$-coherent *frames;*
(iii) $\mathbf{FCP_O}$ *is valid in the class of* $\mathbf{FCP_O}$-permitted *frames;*
(iv) $\mathbf{FCP_P}$ *is valid in the class of* $\mathbf{FCP_P}$-permitted *frames;*
(v) $\mathbf{FCP2_P}$ *is valid in the class of* $\mathbf{FCP2_P}$-permitted *frames;*

Completeness results for the four deontic systems are ensured.

Theorem 4.6

(i) $\mathbf{E}$ *is sound and complete w.r.t. the class of deontic neighbourhood frames;*
(ii) $\mathbf{Min}$ *is sound and complete w.r.t. the class of* $\mathbf{P_s}$*- and* $\mathbf{P_w}$-coherent *frames;*
(iii) $\mathbf{FCP_1}$ *is sound and complete w.r.t. the class of* $\mathbf{FCP_O}$*- and* $\mathbf{FCP_P}$-permitted *frames;*
(iv) $\mathbf{FCP_2}$ *is sound and complete w.r.t. the class of* $\mathbf{P}$-supplemented*,* $\mathbf{FCP_O}$*- and* $\mathbf{FCP_P}$-permitted *frames;*
(v) $\mathbf{FCP_3}$ *is sound and complete w.r.t. the class of* $\mathbf{FCP_O}$*- and* $\mathbf{FCP2_P}$-permitted *frames;*
(vi) $\mathbf{FCP_4}$ *is sound and complete w.r.t. the class of* $\mathbf{P}$-supplemented*,* $\mathbf{FCP_O}$*- and* $\mathbf{FCP2_P}$-permitted *frames.*

Next, a corollary showing the relative strength of the four deontic systems.

Corollary 4.7

(i) $\mathbf{FCP_1} \subset \mathbf{FCP_2} \subset \mathbf{FCP_4}$ *and*
$\mathbf{FCP_1} \subset \mathbf{FCP_3} \subset \mathbf{FCP_4}$.

(ii) *Let* $\mathbf{L_1}, \mathbf{L_2} \in \{\mathbf{FCP_i}, 1 \leq i \leq 4\}$, *and let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be classes of frames adequate for* $\mathbf{L_1}$ *and* $\mathbf{L_2}$. *If* $\mathbf{L_1} \subset \mathbf{L_2}$ *then* $\mathcal{C}_2 \subset \mathcal{C}_1$.

Finally, we are going to examine the issue of decidability. To this end we recall the result by Lewis [18], who proved that every intensional logic that is axiomatisable by axioms that do not contain iterative operators (non-iterative axioms) has the finite model property; A formula (axiom) A is non-iterative iff for every subformula $\Box_i B/\Diamond_i B$ of A, B does not contain a modal operator. It is immediate to verify that the axioms $\mathbf{D_s}$, $\mathbf{D_w}$, $\mathbf{M}$, $\mathbf{FCP_O}$, $\mathbf{FCP_P}$ and $\mathbf{FCP2_P}$ are non-iterative, hence we have the following theorem.

Theorem 4.8 *The logics* $\mathbf{FCP_1}$, $\mathbf{FCP_2}$, $\mathbf{FCP_3}$ *and* $\mathbf{FCP_4}$ *have the finite model property, and hence are decidable.*

5 Conclusions

In this paper we have investigated how, and if the notion of free choice permission is admissible in modal deontic logic. As is well known, several problems can be put forward in regard to this notion, the most fundamental of them being the so-called Permission Explosion Problem, according to which all systems containing **FCP** and closed under **RM** and **RM-P** license the derivation of any arbitrary permission whenever at least one specific permission is true.

We argued (Section 1.1) that a plausible solution to this problem is to jump from monotonic into classical deontic logics, i.e., systems closed under **RE** but not **RM**. This solution does not necessarily mean that the resulting deontic system is very weak, as far as permission is concerned, if further schemata are added (Sections 2.3 and 4.2).

The basic intuitions for extending classical deontic logics are the following:

(i) We assume in background the distinction between norms and obligations/permissions. While we conceptually accept that the normative system may contain conflicting norms, it is logically inadmissible that such norms generate actual conflicting obligations/permissions since conflicts must be rationally solved, otherwise no obligation/permission can be obtained; hence, we validate schemata $\mathbf{D_s}$ and $\mathbf{D_w}$;

(ii) Free choice permission is strong permission, meaning that it is a permission generated by explicit permissive norms;

(iii) The possibility of detaching single strong permissions from disjunctive strong permissions, i.e., $\mathbf{P_s}q$ from $\mathbf{P_s}(p \vee q)$ strictly depends on the fact that $\mathbf{O}\neg p$ is not the case.

Taking the above points into account, we thus proposed different guarded variants of **FCP** that significantly increase the inferential power of the logic. In particular, four Hilbert-style classical deontic systems were presented.

We observed that two of these systems are classical modal systems, while we can have other two acceptable systems which are monotonic. In fact, the fact that those two systems are closed under **RM** does not lead to full Permission Explosion, but only to a "controlled" version of it: indeed, in systems like $\mathbf{FCP_2}$ any permission is obtainable via free choice permission *only if* it is not incompatible with existing prohibitions.

Some directions for future work can be identified. In particular:

- It is still an open issue to fully discuss the Resource Sensitivity Problem in our setting. In fact, while we argued that this problem goes beyond our paper, there are scenarios where our intuitions are relevant for this problem as well. For example, suppose that there is a fruit basket in the kitchen containing a banana and an apple. Bob and Alice are permitted to eat the banana or the apple and Alice first eats the former. Bob cannot do anything but take the apple. However, if Bob is allergic of apples, so no permission can be reasonably derived because it is forbidden for him to eat the apple.
- Our idea of free choice permission relies on the fact that no strong permission can be detached from a disjunctive permissive expression if another norm allows for deriving a conflicting obligation. Hence a full understanding of schemata such as $\mathbf{FCP_O}$ or $\mathbf{FCP_P}$ may benefit for an explicit logical treatment of the logic of norms adopting defeasible reasoning [11]; we plan to investigate how to integrate the approach presented in the paper and the computationally oriented approach offered by Defeasible Deontic Logic [13].

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A Basic Properties of the Deontic Systems

Let us start by proving Lemma 4.5.

Lemma 4.5 *For any deontic neighbourhood frame $\mathcal{F}$,*

(i) $\mathbf{D_s}$ *is valid in the class of* $\mathbf{P_s}$-coherent *frames;*

(ii) $\mathbf{D_w}$ *is valid in the class of* $\mathbf{P_w}$-coherent *frames;*

(iii) $\mathbf{FCP_O}$ *is valid in the class of* $\mathbf{FCP_O}$-permitted *frames;*

(iv) $\mathbf{FCP_P}$ *is valid in the class of* $\mathbf{FCP_P}$-permitted *frames;*

(v) $\mathbf{FCP2_P}$ *is valid in the class of* $\mathbf{FCP2_P}$-permitted *frames;*

Proof. The proof for case (i) is straightforward. The proof of (ii) is trivial and standard. Both are omitted.

Case (iii) – Consider any frame $\mathcal{F}$ that is $\mathbf{FCP_O}$-*permitted* but such that $\mathcal{F} \not\models \mathbf{FCP_O}$. This means that there exists a model $\mathcal{M} = \langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P}, V\rangle$ based on $\mathcal{F}$ such that $\mathcal{M} \not\models \mathbf{FCP_O}$, i.e., there is a world $w \in W$ where

$$\mathcal{M}, w \models \mathbf{P_s}(p \vee q) \wedge \mathbf{O}\neg p \tag{A.1}$$

$$\mathcal{M}, w \not\models \mathbf{P_s} q \tag{A.2}$$

By construction, from (A.2) we have $||q||_{\mathcal{M}} \notin \mathcal{N}_{\mathbf{P}}(w)$, while from (A.1) we have $||p||_{\mathcal{M}} \cup ||q||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{P}}(w)$ and $W - ||p||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{O}}(w)$, so $\mathcal{F}$ is not $\mathbf{FCP_O}$-*permitted*.

Cases (*iv*) *and* (*v*) – The proofs are similar to the one for Case (iii) and are omitted. □

The definitions of some basic notions and of canonical model for the classical bimodal logic **E** (just consisting of **RE** for **O** and $\mathbf{P_s}$) are standard.

In the rest of this section when we refer to a Deontic System **S** we mean one the logic axiomatised in Section 4.

Definition A.1 [**S**-maximality] A set w is maximal iff it is **S**-consistent and for any formula p, either $p \in w$, or $\neg p \in w$.

Lemma A.2 (Lindenbaum's Lemma) *For any Deontic System* **S***, any consistent set w of formulae can be extended to an* **S***-maximal set w^+.*

Definition A.3 [Canonical Model [10, 23]] A canonical neighbourhood model $\mathcal{M} = \langle W, \mathcal{N}_{\mathbf{O}}, \mathcal{N}_{\mathbf{P}}, V\rangle$ for any system **S** in our language $\mathcal{L}$ (where $\mathbf{S} \supseteq \mathbf{E}$) is defined as follows:

(i) W is the set of all the **S**-maximal sets.

(ii) For any propositional letter p, $\|p\|_{\mathcal{M}} := |p|_{\mathbf{S}}$, where $|p|_{\mathbf{S}} := \{w \in W \mid p \in w\}$.

(iii) If $\Box \in \{\mathbf{O}, \mathbf{P_s}\}$, let $\mathcal{N}_{\Box} := \bigcup_{w \in W} \mathcal{N}_{\Box}(w)$ where for each world w, $\mathcal{N}_{\Box}(w) := \{\|a_i\|_{\mathcal{M}} \mid \Box a_i \in w\}$.

Lemma A.4 (Truth Lemma [10, 23]) *If $\mathcal{M} = \langle W, \mathcal{N}_{\mathbf{O}}, \mathcal{N}_{\mathbf{P}}, V\rangle$ is canonical for* **E***, then for any $w \in W$ and for any formula p, $p \in w$ iff $\mathcal{M}, w \models p$.*

Thus, we have as usual basic completeness result for **E**. To cover the other systems, it is enough to prove that all frame properties for the relevant schemata and rules are canonical.

Lemma A.5 *The frame properties of Definition 4.4 are canonical.*

Proof. The proofs for **□-*supplementation***, $\mathbf{P_w}$-***coherence***, and $\mathbf{P_s}$-***coherence*** are standard.

$\mathbf{FCP_P}$-***permission*** – Let us consider a canonical model $\mathcal{M}$ for $\mathbf{FCP_P}$, any world w in it, and any truth sets such that $||p||_{\mathcal{M}} \cup ||q||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{P}}(w)$ and $W - ||q||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{O}}(w)$. Clearly, $||p \vee q||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{P}}(w)$. Since $\mathbf{FCP_P}$ is valid (Lemma 4.5), then $\mathbf{P_s}p \in w$. By construction, this means that $||p||_{\mathcal{M}} \in \mathcal{N}_{\mathbf{P}}(w)$, thus the model is $\mathbf{FCP_P}$-permitted.

$\mathbf{FCP_O}$-***permission*** and $\mathbf{FCP2_P}$-***permission***– Similar to the case above.
□

Hence, the following result is ensured.

Theorem 4.6

(i) **E** *is sound and complete w.r.t. the class of deontic neighbourhood frames;*

(ii) $\mathbf{Min}$ *is sound and complete w.r.t. the class of* $\mathbf{P_s}$- *and* $\mathbf{P_w}$-coherent *frames;*

(iii) $\mathbf{FCP_1}$ *is sound and complete w.r.t. the class of* $\mathbf{FCP_O}$- *and* $\mathbf{FCP_P}$-permitted *frames;*

(iv) $\mathbf{FCP_2}$ *is sound and complete w.r.t. the class of* $\mathbf{P}$-supplemented*,* $\mathbf{FCP_O}$- *and* $\mathbf{FCP_P}$-permitted *frames;*

(v) $\mathbf{FCP_3}$ *is sound and complete w.r.t. the class of* $\mathbf{FCP_O}$- *and* $\mathbf{FCP2_P}$-permitted *frames;*

(vi) $\mathbf{FCP_4}$ *is sound and complete w.r.t. the class of* $\mathbf{P}$-supplemented*,* $\mathbf{FCP_O}$- *and* $\mathbf{FCP2_P}$-permitted *frames.*

Finally, let us prove Corollary 4.7.

Corollary 4.7

(i) $\mathbf{FCP_1} \subset \mathbf{FCP_2} \subset \mathbf{FCP_4}$ *and*
$\mathbf{FCP_1} \subset \mathbf{FCP_3} \subset \mathbf{FCP_4}$.

(ii) *Let* $\mathbf{L_1}, \mathbf{L_2} \in \{\mathbf{FCP_i}, 1 \leq i \leq 4\}$*, and let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be classes of frames adequate for* $\mathbf{L_1}$ *and* $\mathbf{L_2}$*. If* $\mathbf{L_1} \subset \mathbf{L_2}$ *then* $\mathcal{C}_2 \subset \mathcal{C}_1$*.*

Proof. *Case* (i) – For $\mathbf{FCP_1} \subset \mathbf{FCP_2}$ the inclusion is trivial given that every axiom of $\mathbf{FCP_1}$ is also an axiom of $\mathbf{FCP_2}$. To show that the inlcusion is strict consider the model $\mathcal{M} = \langle W, \mathcal{N}_\mathbf{O}, \mathcal{N}_\mathbf{P}, V\rangle$, where:

- $W = \{w_1, w_2, w_3, w_4, w_5\}$;
- $V(a) = \{w_1, w_4, w_5\}$, $V(b) = \{w_2, w_3, w_4\}$ and $V(c) = \{w_1, w_2\}$;
- $\mathcal{N}_\mathbf{O}(w_1) = \{\{w_4\}\}$; and
- $\mathcal{N}_\mathbf{P}(w_1) = \{\{w_1, w_2, w_3\}, \{w_1, w_2\}\}$.

It is easy to verify that the model is $\mathbf{FCP_O}$-permitted, $\mathbf{P_s}(\neg a \vee c)$ and $\mathbf{O}(a \wedge c)$ are true in w_1: $||\neg a \vee c||_\mathcal{M} = \{w_1, w_2, w_3\} \in \mathcal{N}_\mathbf{P}(w_1)$ and $||a \wedge c||_\mathcal{M} = \{w_4\} \in \mathcal{N}_\mathbf{O}(w_1)$. However, the model is not $\mathbf{O}$-supplemented: $||a \wedge c||_\mathcal{M} = \{w_4\} \in \mathcal{N}_\mathbf{O}(w_1)$, $\{w_4\} = ||a||_\mathcal{M} \cap ||c||_\mathcal{M}$, but $||a||_\mathcal{M}, ||c||_\mathcal{M} \notin \mathcal{N}_\mathbf{O}(w_1)$, falsifying the following instance of $\mathbf{M}$: $\mathbf{O}(a \wedge c) \rightarrow \mathbf{O}a \wedge \mathbf{O}c$.

For $\mathbf{FCP_1} \subset \mathbf{FCP_3}$ it is immediate to verify that $\mathbf{FCP2_P}$ implies $\mathbf{FCP_P}$ in $\mathbf{CPL}$ but not the other way around, and the same relationship holds for the corresponding semantic conditions.

For $\mathbf{FCP_2} \subset \mathbf{FCP_4}$, the result follows from $\mathbf{FCP_1} \subset \mathbf{FCP_3}$.

For $\mathbf{FCP_3} \subset \mathbf{FCP_4}$, the inclusion is trivial and we can reuse the model to show the strictness of the inclusion between $\mathbf{FCP_1}$ and $\mathbf{FCP_2}$.

Case (*ii*) – The result follows from Case (i) above and Theorem 4.6. □